# Integrability of classical and semiclassical derivative non-linear Schrodinger equation with non-ultralocal canonical structure



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# Integrability of classical and semiclassical derivative non-linear Schrödinger equation with non-ultralocal canonical structure

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**Abstract.** The derivative non-linear Schrödinger equation exhibiting non-ultralocal canonical structure is investigated to obtain various relationships including classical $r$-$s$ matrices and the Yang-Baxter relation modified by the non-ultralocality. The complete integrability of the system is established through explicit action-angle canonical variables. An attempt has been made to solve the corresponding quantum field model in the semiclassical approximation. A possibility of eliminating non-ultralocality of the model, necessary for exact quantum inverse scattering treatment, is demonstrated.

## 1. Introduction

In the last two decades a number of integrable classical non-linear field models have been discovered in two spacetime dimensions and inverse scattering theory (IST) has been successfully applied to them to explore exact analytic solutions [1, 2]. IST was further generalised (QIST) to cover quantum field-theoretic versions of some classically integrable systems, such as the non-linear Schrödinger equation (NLS) [3, 4], the sine-Gordon model [5], the quantum three-wave interaction [6, 7] and various other models [8, 9]. Recent years have witnessed a significant increase in research activities on QIST and remarkable progress has been achieved in this field [8, 9]. Some of these achievements are the formulation of the QIST scheme through discrete and continuous methods, reproducing results of a $\delta$-function Bose gas showing interesting connections between QIST and the Bethe ansatz, which reveals a more complete insight of quantum the integrable systems; finding the mass spectrum and $S$ matrix in the SG model, etc. On the other hand, it is surprising to observe that, in spite of the impressive progress of QIST and extensive research in this field, attention has mostly focused on a particular class of problems, e.g., on integrable systems with ultralocal Poisson bracket relations, i.e. systems with canonical brackets containing only the $\delta$-function without any derivative terms. However, there exists a large class of interesting non-linear systems with non-ultralocal properties, such as KdV, MKdV [1], the complex sine-Gordon [10, 11], the non-linear $\sigma$-model [12], the chiral model [13] and the derivative [14] and modified derivative [15] non-linear Schrödinger equation, etc, for which QIST is not developed and very few works are devoted to this challenging problem. Tsyplyaev [16] was possibly the first to initiate this problem and to find a new method for calculating the classical $r$ matrix for such non-ultralocal models. Thereafter, but only very recently, Maillet [10, 12] has shown a distinct approach in this field by introducing $r$ and $s$ matrices and found a new Yang-Baxter relation along with some other important results. There was also a recent attempt to tackle the KdV equation by transforming its canonical structure [17]. Unfortunately, however, most of these methods were confined to the classical models and to date there is no clear prescription for handling the corresponding quantum versions.

The aim of our present study is to elaborate this method for the derivative non-linear Schrödinger equation (DNLS) and extend it further to solve the quantum model in a semiclassical approximation. We construct $r$ and $s$ matrices [10], deduce a modified classical Yang–Baxter equation using the Jacobi identity method developed in [10] and check different general formulations for DNLS. We also find, as a byproduct, the Poisson bracket relation between scattering data leading to action-angle variables, which enable us to show the complete integrability of the classical model. In the semiclassical approximation the quantum Bethe state with the corresponding eigenvalues has been obtained for the infinite conserved quantities. The formation of a collective bound state (soliton) is analysed. A possibility of eliminating non-ultralocality through gauge transformation is demonstrated.

The organisation of the paper is as follows. In § 2 we derive different relevant formulae for non-ultralocal DNLS. Section 3 establishes complete integrability of the classical DNLS. Section 4 describes a semiclassical approach to solve the quantum model. Section 5 presents a way to free the model from non-ultralocality. Section 6 is the concluding section.

## 2. Non-ultralocal structures related to DNLS

It is known that an integrable non-linear PDE may be represented by the Lax pair, which again may be replaced only by the spectral operator $L$ and the Poisson bracket relation. For DNLS [14]

$$\mathrm{i}\psi_t + \psi_{xx} + \mathrm{i}\varkappa(\psi^+\psi\psi)_x = 0 \tag{2.1}$$

we have

$$L(x, \lambda) = -\mathrm{i}\lambda^2\sigma^3 + \lambda\varkappa\psi^+(x)\sigma^- - \lambda\psi(x)\sigma^+ \tag{2.2}$$

where $\sigma^3$, $\sigma^\pm$ are Pauli matrices and the Poisson bracket relation

$$\{\psi(x), \psi^+(y)\} = \partial_x\delta(x-y) \tag{2.3}$$

with non-ultralocal structure. Direct calculation in this case leads to

$$\{L(x, \lambda) \otimes L(y, \mu)\} = B\partial_x\delta(x-y) + C\partial_y\delta(x-y) \tag{2.4}$$

where $B = -\varkappa\lambda\mu\sigma^+ \otimes \sigma^-$, $C = \varkappa\lambda\mu\sigma^- \otimes \sigma^+$ and a notation $(A \otimes B)_{ij,kl} = A_{ik}B_{jl}$, $\{A \otimes B\}_{ij,kl} = \{A_{ik}, B_{jl}\}$ has been introduced. The monodromy matrix satisfies the equations [4, 13]:

$$\partial_z\tau(z, y, \lambda) = L(z, \lambda)\tau(z, y, \lambda) \tag{2.5a}$$

$$\partial_z\tau(x, z, \lambda) = -\tau(x, z, \lambda)L(z, \lambda) \tag{2.5b}$$

and is defined at various limits as

$$\tau_+(y, \lambda) = E_+^{-1}(x, \lambda)\tau(x, y, \lambda)\big|_{x\to+\infty} \qquad \tau_+(y, \lambda)\big|_{y\to+\infty} = E_+^{-1}(y, \lambda). \tag{2.6a}$$

$$\tau_-(x, \lambda) = \tau(x, y, \lambda)E_-(y, \lambda)\big|_{y\to-\infty} \qquad \tau_-(x, \lambda)\big|_{x\to-\infty} = E_-(x, \lambda) \tag{2.6b}$$

and

$$\tau(\lambda) = E_+^{-1}(x, \lambda)\tau(x, y, \lambda)E_-(y, \lambda)|_{\substack{x\to+\infty \\ y\to-\infty}} = \tau_+(z, \lambda)\tau_-(z, \lambda). \tag{2.6c}$$

In the case of DNLS (2.2) with vanishing boundary condition on the field $|\psi|_{|x|\to\infty} \to 0$, one gets $E_\pm(x, \lambda) = \exp(-i\lambda^2\sigma^3 x)|_{x\to\pm\infty}$. We may now calculate the corresponding Poisson bracket [12] for our case:

$$\begin{aligned}\{\tau(\lambda)\otimes\tau(\mu)\} &= \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dz'\, \tau_+(z, \lambda)\otimes\tau_+(z', \mu) \\ &\quad \times\{L(z, \lambda)\otimes L(z', \mu)\}\tau_-(z, \lambda)\otimes\tau_-(z', \mu) \\ &= \int_{-\infty}^{\infty} dz\, \tau_+(z, \lambda)\otimes\tau_+(z, \mu)\Omega\tau_-(z, \lambda)\otimes\tau_-(z, \mu)\end{aligned} \tag{2.7a}$$

where

$$\Omega = [B, L(z, \lambda)\otimes 1] + [C, 1\otimes L(z, \mu)]. \tag{2.7b}$$

In deducing (2.7) we have successively used definition (2.6), relation (2.4), equation (2.5) and the matrix property $AB\otimes CD = (A\otimes C)(B\otimes D)$. Assuming now the existence of a $r_0(\lambda, \mu, z)$ matrix such that

$$\tau_+\otimes\tau'_+\Omega\tau_-\otimes\tau'_- = \partial_z(\tau_+\otimes\tau'_+ r_0\tau_-\otimes\tau'_-) \tag{2.8}$$

where a prime denotes different arguments $\lambda$ in parameter space, on one hand, we may integrate (2.7) to obtain

$$\{\tau\otimes\tau'\} = r_+\tau(\lambda)\otimes\tau(\mu) - \tau(\lambda)\otimes\tau(\mu)r_- \tag{2.9a}$$

where

$$r_\pm(\lambda, \mu) = \lim_{z\to\pm\infty} (E_\pm^{-1}(z, \lambda)\otimes E_\pm^{-1}(z, \mu)r_0(\lambda, \mu, z)E_\pm(z, \lambda)\otimes E_\pm(z, \mu)) \tag{2.9b}$$

is introduced and, on the other hand, from (2.8) using (2.5) we derive the relation [16]:

$$\partial_x r_0(\lambda, \mu, x) + [r_0, L\otimes 1 + 1\otimes L'] = \Omega. \tag{2.10}$$

In a more general case

$$\{L(z, \lambda)\otimes L(z', \mu)\} = A\delta(z-z') + B\partial_z\delta(z-z') + C\partial_{z'}\delta(z-z') \tag{2.4'}$$

where $A = A(z, \lambda, \mu)$, $B = B(z, z', \lambda, \mu)$ and $C = C(z, z', \lambda, \mu)$ are different functions of field variables, a similar treatment leads to the relation (2.10) with $\Omega$ given by [13]:

$$\Omega = A + [L\otimes 1, B] - [1\otimes L', C] + \partial_z(B + C). \tag{2.7b'}$$

Note that putting $A = 0$, $\partial_z B = 0$, $\partial_z C = 0$ in (2.4') and (2.7$b'$) we recover (2.4) and (2.7) for DNLS and for this case $r$ and $s$ matrices introduced in [13] take the following form:

$$s = -\tfrac{1}{2}(B - C) = \tfrac{1}{2}\varkappa\lambda\mu(\sigma^-\otimes\sigma^+ + \sigma^+\otimes\sigma^-) \tag{2.11a}$$

$$r = r_0 + \tfrac{1}{2}(B + C) = r_0 + \tfrac{1}{2}\varkappa\lambda\mu(\sigma^-\otimes\sigma^+ - \sigma^+\otimes\sigma^-). \tag{2.11b}$$

Note that in such models, in general, additional boundary terms related to non-ultralocality should appear on the right-hand side of (2.9$a$) [10]. But since for DNLS, matrices like $B(\lambda, \mu)$ and $C(\lambda, \mu)$ do not have any poles at $\lambda^2 \to \mu^2$ corresponding matrices $B_\pm(\lambda, \mu)$ and $C_\pm(\lambda, \mu)$ vanish at the infinite interval limit giving no additional contribution to Poisson bracket (2.9$a$).

One may also check that $PBP=-C$, $PsP=s$ and $Pr(\lambda,\mu)P=-r(\mu,\lambda)$, provided $Pr_0(\lambda,\mu)P=-r_0(\mu,\lambda)$, which is clear from (2.13). We find $r+s=r_0-\varkappa\lambda\mu\sigma^+\otimes\sigma^-$ and $r-s=r_0+\varkappa\lambda\mu\sigma^-\otimes\sigma^+$, which leads through the Jacobi identity [13] of (2.7) to the modified Yang-Baxter relation

$$\begin{aligned}\{[r_{023}(\mu,\eta),r_{012}(\lambda,\mu)]&+[r_{023}(\mu,\eta),r_{013}(\lambda,\eta)]\\&+[r_{013}(\lambda,\eta),r_{012}(\lambda,\mu)]\}+D=0\end{aligned}\tag{2.12a}$$

where the first part is the standard classical Yang-Baxter equation [8], while

$$\begin{aligned}D=\varkappa\{[r_{023}(\mu,\eta),\sigma^-\otimes\sigma^+\otimes\mathbb{1}]\lambda\mu&\\&+[\mathbb{1}\otimes\sigma^-\otimes\sigma^+,(r_{012}(\lambda,\mu)+r_{013}(\lambda,\eta))]\mu\eta\\&+[\sigma^+\otimes\sigma^-\otimes\mathbb{1},r_{013}(\lambda,\eta)]\lambda\mu\\&+[\sigma^-\otimes\mathbb{1}\otimes\sigma^+,(r_{012}(\lambda,\mu)-r_{013}(\mu,\eta))]\lambda\eta\\&+\varkappa\lambda\mu\eta(\lambda\sigma^3\otimes\sigma^-\otimes\sigma^+-\mu\sigma^-\otimes\sigma^3\otimes\sigma^+)\}\end{aligned}\tag{2.12b}$$

is the contribution due to non-ultralocality. In (2.12) we have assumed the locality condition $\partial_x r_0=0$, which solves (2.10) after a rather lengthy but straightforward calculation to yield

$$r_0(\lambda,\mu)=-(\lambda^2-\mu^2)^{-1}(\tfrac{1}{2}\varkappa(\lambda\mu)^2\sigma^3\otimes\sigma^3-\lambda^2B+\mu^2C).\tag{2.13}$$

In the limit of an infinitely large interval using (2.9) and regularising the singularities through principal values, we finally obtain

$$r_\pm=-\varkappa\left(\frac{\xi\zeta}{2(\zeta-\xi)}\sigma^3\otimes\sigma^3\pm2\mathrm{i}\pi\zeta^2\delta(\zeta-\xi)(\sigma^+\otimes\sigma^--\sigma^-\otimes\sigma^+)\right)\tag{2.14}$$

where $\zeta=\lambda^2$ and $\xi=\mu^2$.

## 3. Complete integrability of classical DNLS

For evaluating the Poisson brackets between the elements of the scattering matrix $\tau(\zeta)$, let

$$\tau(\zeta)=\begin{pmatrix}a(\zeta), & b(\zeta)\\ -\bar{b}(\zeta), & \bar{a}(\zeta)\end{pmatrix}$$

which from (2.14) and (2.9$a$) yield

$$\{a(\zeta),a(\xi)\}=\{a(\zeta),\bar{a}(\xi)\}=\{b(\zeta),b(\xi)\}=0\tag{3.1a}$$

$$\{a(\zeta),b(\xi)\}=c(\zeta,\xi)a(\zeta)b(\xi)+d(\zeta)b(\zeta)a(\xi)\delta(\zeta-\xi)\tag{3.1b}$$

$$\{a(\zeta),\bar{b}(\xi)\}=-c(\zeta,\xi)a(\zeta)\bar{b}(\xi)-d(\zeta)\bar{b}(\zeta)a(\xi)\delta(\zeta-\xi)\tag{3.1c}$$

$$\{b(\zeta),\bar{b}(\xi)\}=2d(\zeta)|a(\zeta)|^2\delta(\zeta-\xi)\tag{3.1d}$$

etc, where $c(\zeta,\xi)=-\varkappa\zeta\xi/(\zeta-\xi)$ and $d(\zeta)=2\mathrm{i}\zeta^2\varkappa\pi$. Note that, since $a(\zeta)$ is analytic in the upper half $\zeta$-plane [14], the $\delta$-function part in (3.1$b$) may also be formally generated from the singularity of $c(\zeta,\xi)$ at $\zeta=\xi$ by replacing $(\zeta-\xi)^{-1}\rightarrow(\zeta-\xi+\mathrm{i}\varepsilon)^{-1}|_{\varepsilon\rightarrow0^+}$ which, clearly, vanishes for $\zeta\neq\xi$. Therefore, expanding

$$\ln a(\zeta)=\mathrm{i}\varkappa\sum_{n=0}^{\infty}c_n/(2\zeta)^n\qquad c(\zeta,\xi)=-\varkappa\sum_{n=0}^{\infty}\xi^{n+1}\zeta^{-n}$$

one finds

$$(\mathrm{i}/2^n)\{c_n, b(\xi)\} = -\xi^{n+1} b(\xi) \qquad \{c_n, c_m\} = 0$$

or for Hamiltonian $H = 2C_1$ the evolution equation

$$\dot{b}(\xi) = \{H, b(\xi)\} = 4\mathrm{i}\xi^2 b(\xi) \qquad \dot{a}(\xi) = \{H, a(\xi)\} = 0 \tag{3.2}$$

yielding the solution

$$b(\xi, t) = b(\xi, 0)\exp(4\mathrm{i}\xi^2 t) \qquad a(\xi, t) = a(\xi, 0).$$

Representing [14]

$$a(\zeta) = \exp(\mathrm{i}\mu_0)\prod_{k=1}^{N}\left(\frac{\zeta - \zeta_k}{\zeta - \zeta_k^*}\right)\exp\left(\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\mathrm{d}\zeta'}{\zeta' - \zeta}\ln(|a(\zeta')|^2)\right) \tag{3.3}$$

where $\mu_0 = -\frac{1}{2}\varkappa\int_{-\infty}^{\infty}(\psi^+\psi)\,\mathrm{d}x$, the infinite conserved laws may be given by

$$c_0 = \frac{\mu_0}{\varkappa} \qquad c_n = \frac{-\mathrm{i}2^n}{\varkappa n}\sum_{k=1}^{N}(\zeta_k^{*n} - \zeta_k^n) + \frac{1}{2\varkappa\pi}\int_{-\infty}^{\infty}(2\zeta)^{n-1}\ln(|a|^2)\,\mathrm{d}(2\zeta). \tag{3.4}$$

Now separating real and imaginary parts of (3.1$b$), after a little manipulation, one obtains

$$n(\zeta) = -\frac{1}{\varkappa\pi\zeta^2}\ln|a(\zeta)| \qquad \varphi(\xi) = \arg b(\xi) \tag{3.5}$$

with the Poisson bracket relations

$$\{n(\zeta), \varphi(\xi)\} = \delta(\zeta - \xi) \qquad \{n(\zeta), n(\xi)\} = \{\varphi(\zeta), \varphi(\xi)\} = 0 \tag{3.6}$$

as the canonical action-angle variables for the continuum spectrum and through the representation (3.3)

$$N_j = -\frac{1}{3\varkappa}\frac{1}{\zeta_j} \qquad N_j^0 = -\frac{\mathrm{i}}{\varkappa}\frac{\delta\mu_0}{\delta\zeta_j} \qquad \varphi_l = \ln b_l \tag{3.7}$$

with relations $\{N_j, \varphi_l\} = \delta_{jl}$, $\{N_i, N_j\} = \{\varphi_i, \varphi_j\} = 0$ as action-angles for a discrete spectrum. The infinite conserved quantities (3.4) may be clearly expressed through action variables only, in the form

$$C_0 = \frac{\mathrm{i}}{3\varkappa}\sum_{j=1}^{N}\int N_j^0 N_j^{-2}\delta N_j$$
$$C_n = -\frac{\mathrm{i}}{\varkappa n}\left(-\frac{2}{3\varkappa}\right)^n\left(\sum_{k=1}^{N}(N_k^{*-n} - N_k^{-n})\right) - 2^{n-2}\int_{-\infty}^{\infty}\zeta'^{n+1}n(\zeta')\,\mathrm{d}\zeta'. \tag{3.8}$$

Due to the trivial Poisson brackets between action variables the infinite set of conserved quantities are in involution: $\{C_n, C_m\} = 0$. We now have the Hamiltonian also expressed only through action variables as

$$H = -8C_1 = -\frac{16\mathrm{i}}{3\varkappa^2}\left(\sum_{k=1}^{N}(N_k^{*-1} - N_k^{-1})\right) + 8\int_{-\infty}^{\infty}\zeta'^2 n(\zeta')\,\mathrm{d}\zeta'$$
$$= \frac{16}{3\varkappa^2}\sum_{k=1}^{N}\rho_k^2\sin\alpha_k + 8\int_{-\infty}^{\infty}\zeta'^2 n(\zeta')\,\mathrm{d}\zeta' \tag{3.9}$$

establishing the complete integrability [1] of the classical DNLS, where we have set $N_k = \rho_k^{-2}\mathrm{e}^{\mathrm{i}\alpha_k}$, $0 \leq \alpha_k \leq \pi$. Note that the first part of the spectrum (3.9) corresponds to the soliton contribution, which vanishes for $\alpha_k = 0$, i.e. for $\mathrm{Im}\,\lambda_k = 0$, leaving only the continuum spectrum related to radiation.

## 4. Semiclassical quantisation of DNLS

It is interesting to note that, although the canonical field variables for DNLS exhibit non-ultralocality in the configuration space, the action-angle variables are completely ultralocal in the parameter space and have standard canonical relations. This observation along with the known fact that the exact quantum and semiclassical treatments yield the same result for NLS [18] motivates us to propose a symmetrised semiclassical method in analogy with the exact quantum treatment. We define

$$\begin{aligned} R_\pm(\zeta,\xi) &= P(\mathbb{1}\otimes\mathbb{1} - \mathrm{i}\hbar r_\pm(\zeta,\xi)) \\ &= P - \tfrac{1}{2}\mathrm{i}\hbar[C(\zeta,\xi)(\mathbb{1}\otimes\mathbb{1}+\sigma^3\otimes\sigma^3) \\ &\quad \pm d(\zeta)\delta(\zeta-\xi)(\mathbb{1}\otimes\sigma^3-\sigma^3\otimes\mathbb{1})] \end{aligned} \tag{4.1}$$

and replacing the Poisson bracket by the commutator we readily obtain from (2.9$a$) the famous relation [8] (for the infinite interval):

$$R_+(\zeta,\xi)T(\lambda)\otimes T(\mu) = T(\mu)\otimes T(\lambda)R_-(\zeta,\xi) \tag{4.2}$$

reflecting the quantum integrability in the semiclassical limit. Equation (4.2) leads to the following commutation relations:

$$[A(\zeta), A(\xi)] = [A(\zeta), A^+(\xi)] = [B(\zeta), B(\xi)] = 0 \tag{4.3a}$$

$$A(\zeta)B^+(\xi) = \bar{a}(\zeta,\xi)B^+(\xi)A(\zeta) + b(\zeta)\delta(\zeta-\xi)A(\zeta)B^+(\xi) \tag{4.3b}$$

$$[B(\zeta), B^+(\xi)] = -2b(\zeta)A^+(\zeta)A(\zeta)\delta(\zeta-\xi) \tag{4.3c}$$

where $b(\zeta) = 2\hbar\pi\varkappa\zeta^2$ and $a(\zeta,\xi) = (1+\mathrm{i}\hbar c(\zeta,\xi))/(1-\mathrm{i}\hbar c(\zeta,\xi))$. Definining vacuum $|0\rangle$ as $B(\zeta)|0\rangle = 0$ and $A(\zeta)|0\rangle = |0\rangle$ we may construct the $N$-particle eigenstate as $|\Phi_N\rangle = \Pi_{j=1}^N B^+(\zeta_j)|0\rangle$, which gives clearly the operator action

$$\begin{aligned} \ln A(\zeta)|\Phi_N\rangle &= \ln \bar{a}_N|\Phi_N\rangle \\ &= \sum_{j=1}^N \ln \bar{a}(\zeta,\xi_j)|\Phi_N\rangle \\ &= \sum_{j=1}^N \ln\{(\bar{P}_j/P_j)[(\zeta-P_j)/(\zeta-\bar{P}_j)]\}|\Phi_N\rangle. \end{aligned} \tag{4.4}$$

Assuming a similar expansion for operators [4]:

$$\ln A(\zeta) = \mathrm{i}\varkappa\hbar \sum_{m=0}^\infty \hat{A}_m\zeta^{-m} \tag{4.5}$$

and defining $\hat{\xi}_j =: \xi_j:$ we obtain the operator relation

$$\begin{aligned} \hat{A}_0 &= \frac{2}{\varkappa\hbar}\sum_{j=1}^N \beta_j \\ \hat{A}_m &= -\frac{\mathrm{i}}{\varkappa\hbar}\sum_{j=1}^N \frac{(-1)^m}{m}[\hat{P}_j^m - \hat{\bar{P}}_j^m] \end{aligned} \tag{4.6}$$

with

$$\hat{P}_j = \hat{\xi}_j(1+\mathrm{i}\hbar\varkappa\hat{\xi}_j)^{-1} = \hat{\mu}_j\,\mathrm{e}^{-\mathrm{i}\hat{\beta}_j} \qquad \hat{\beta}_j = \tan^{-1}(\hbar\varkappa\hat{\xi}_j)$$

and

$$\hat{\mu}_j = \hat{\xi}_j/(1+\hbar^2\varkappa^2\hat{\xi}_j^2)^{1/2} = \frac{1}{\varkappa\hbar}\sin\hat{\beta}_j.$$

Here, due to the complicated structure of (4.6) and contrary to the NLS case, physical interpretations become rather difficult. Nevertheless, the energy eigenvalue of the $N$-particle state given by $H=4C_1$ where

$$\ln \bar{a}_N = \mathrm{i}\varkappa\hbar \sum_{n=1}^{\infty} \frac{C_n^{(N)}}{(2\zeta)^n}$$

leads to

$$H=\frac{8\mathrm{i}}{\varkappa\hbar}\sum_{j=1}^{N}(P_j-\bar{P}_j)=\frac{16}{\varkappa^2\hbar^2}\sum_{j=1}^{N}\sin^2\beta_j=\frac{16}{\varkappa\hbar}\sum_{j=1}^{N}\mu_j\sin\beta_j. \tag{4.7}$$

Note that the spectrum of the $N$-particle scattering state (4.7) is similar to the Hamiltonian for the classical $N$-soliton solution given through action variables by the first term in (3.9).

For finding the bound state or collective soliton state formed by $N$ particles we proceed in analogy with the NLS case [4] and observe that a distribution

$$\xi_j=[\xi_0^{-1}-\mathrm{i}(N+1-2j)\varkappa\hbar]^{-1} \qquad \xi_0=\text{real constant} \tag{4.8}$$

cancels all zeros of $\bar{a}_N$ leaving only a single one corresponding to the one-soliton state:

$$\bar{a}_N=[(\zeta-P_N)/(\zeta-\bar{P}_N)]\prod_{j=1}^{N}\mathrm{e}^{2\mathrm{i}\beta_j} \tag{4.9}$$

where $P_N=(\xi_0^{-1}+\mathrm{i}\varkappa N\hbar)^{-1}$. Hence the energy of the bound state may be given by (4.7), with the values of $\xi_j$ as (4.8) resulting in an interesting expression:

$$H_{\mathrm{sol}}^{(N)}=\frac{16N}{\hbar^2\varkappa^2}\sin^2\beta^{(N)} \qquad \beta^{(N)}=\tan^{-1}(\hbar\varkappa N\xi_0). \tag{4.10}$$

Therefore, the binding energy of this $N$-particle state is given by

$$\begin{aligned}E_{\mathrm{B}}^{(N)}&=NH_{\mathrm{sol}}^{(1)}-H_{\mathrm{sol}}^{(N)}\\&=16N\xi_0^2[(1+\hbar^2\varkappa^2\xi_0^2)^{-1}-(1+\hbar^2\varkappa^2N^2\xi_0^2)^{-1}]\\&\approx 16\hbar\varkappa^2\xi_0^4(N^2-1)N+\mathrm{O}(\hbar^4)\geqslant 0\end{aligned} \tag{4.11}$$

which establishes the required stability of the state. It is interesting also to observe that in the weak-coupling limit the binding energy (4.11) has some resemblance to that of an attractive $\delta$-function Bose gas [3]. In the same limit we also have

$$H_{\mathrm{sol}}^{(N)}\approx 16N\xi_0^2-16\hbar^2\varkappa^2N^3\xi_0^4+\mathrm{O}(\hbar^4)$$

where the first term may be interpreted as the total energy of $N$ free non-interacting ($\varkappa=0$) particles and the rest of the terms are due to interactions.

## 5. Transformation to ultralocality

For an exact QIST treatment one should have an ultralocal model from the very beginning, since at present the QIST formulation is available only for this case. The underlying idea is that a gauge transformation of the form $\tilde{\Phi}=h\Phi$ leading to

$$\tilde{L}=hLh^{-1}+h_xh^{-1} \tag{5.1}$$

might change the non-ultralocal canonical structure to the ultralocal form in suitably chosen variables, which in fact play the role of fundamental canonical variables. Redefinining $q=-\psi/\sqrt{\varkappa}$, $q^+=-\sqrt{\varkappa}\psi^+$ we have $L=-\mathrm{i}\lambda^2\sigma^3+\mathrm{i}\sqrt{\varkappa}(a\sigma^2+b\sigma^1)\lambda$ with $q=a+\mathrm{i}b$. Choosing now the gauge $h=\exp(-\mathrm{i}\sqrt{\varkappa}\lambda\sigma^1\varphi)$, where $\varphi_x=\frac{1}{2}b$, one obtains the transformed matrix $\tilde{L}$ in the form

$$\tilde{L}=-\mathrm{i}\lambda(\lambda\cos\theta-\sqrt{\varkappa}a(x)\sin\theta)\sigma^3+\mathrm{i}\lambda(\lambda\sin\theta+\sqrt{\varkappa}a(x)\cos\theta)\sigma^2 \tag{5.2}$$

where $\theta=\sqrt{\varkappa}\lambda\varphi$, with the ultralocal canonical structure

$$\{\varphi(x), a(y)\}=\mathrm{i}\delta(x-y). \tag{5.3}$$

In deducing (5.2) we have used the identities

$$h\sigma^1h^{-1}=\sigma^1 \qquad h\sigma^{3(2)}h^{-1}=\sigma^{3(2)}\cos\theta(\mp)\sigma^{2(3)}\sin\theta.$$

Proceeding in the same way as discussed in § 2 we obtain

$$\{\tilde{L}(x,\lambda)\otimes\tilde{L}(y,\mu)\}=A(x,\lambda,\mu)\delta(x-y) \tag{5.4}$$

where

$$\begin{aligned}A(x,\lambda,\mu)=-\mathrm{i}\varkappa\lambda\mu[&\alpha(\lambda,\mu,\varphi,a)\sigma^3\otimes\sigma^3+\beta(\lambda,\mu,\varphi,a)\sigma^2\otimes\sigma^2\\&+\gamma(\lambda,\mu,\varphi,a)\sigma^3\otimes\sigma^2-\gamma(\mu,\lambda,\varphi,a)\sigma^2\otimes\sigma^3]\end{aligned} \tag{5.5}$$

with

$$\alpha(\lambda,\mu,\varphi,a)=(\lambda^2-\mu^2)\sin\theta\sin\theta'+\sqrt{\varkappa}(\lambda\cos\theta\cos\theta'-\mu\cos\theta'\sin\theta)a(x)$$

$$\beta(\lambda,\mu,\varphi,a)=(\lambda^2-\mu^2)\cos\theta\cos\theta'+\sqrt{\varkappa}(\lambda\sin\theta\cos\theta'-\mu\sin\theta'\cos\theta)a(x)$$

and

$$\gamma(\lambda,\mu,\varphi,a)=(\lambda^2-\mu^2)\sin\theta\cos\theta'+\sqrt{\varkappa}(\lambda\cos\theta\cos\theta'+\mu\sin\theta\sin\theta')a(x).$$

Note that the form like (5.4) is recovered from the general relation (2.4′) by neglecting the non-ultralocal terms: $B=C=0$, which yields from (2.7$b'$) $\Omega=A$ and from (2.10) the required equation for $r_0(\lambda,\mu,x)$ as

$$\partial_x r_0+[r_0,\tilde{L}\otimes\mathbb{1}+\mathbb{1}\otimes\tilde{L}']=A. \tag{5.6}$$

Unfortunately, we are unable to find yet an explicit form of $r_0$ as a solution of (5.6) and therefore cannot investigate further the integrability properties of the model at the classical level. However, we hope that, due to its ultralocal canonical structure, a direct quantum inverse scattering treatment might be possible.

## 6. Conclusion

The derivative non-linear Schrödinger equation with non-ultralocal canonical structure is investigated classically and various relevant relationships and expressions are extracted. Using this result complete integrability of the system is shown with explicit action-angle variables. A semiclassical treatment revealed the existence of $N$-particle and soliton bound states. This also demonstrates the stability of the bound states and that in the weak-coupling limit the binding energy resembles that of a $\delta$-function Bose gas. Through a suitable gauge transformation the non-ultralocality is removed from the model. It is hoped that this will help in pursuing QIST of this model. This problem is now under investigation and will be reported elsewhere. We also wish to comment in this connection that we are able through similar gauge transformation to free the models like MKdV [19] and sine-Gordon (in light-cone coordinates) [20] from non-ultralocality and solve their quantum version exactly through QIST. We should note, however, that gauge-transformed models, though gauge equivalent, may not be physically the same as the original model [21] and, moreover, for the quantum models this is still an open problem. It is also interesting to observe that for models with a known exact QIST solution, the semiclassical approximation adopted here yields exact results apart from some renormalisation of the coupling constant [19]. If this is supposed to be true for all integrable systems, then even semiclassical DNLS results obtained here might be significant.

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